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On spin-squeezed states and their application to semi-classical kink dynamics in magnetic chains

H Frahm† and J A Hołyst‡

Institut für Theoretische Physik, Universität Hannover, D-3000 Hannover 1, Federal Republic of Germany

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Abstract. We introduce an extension of the Radcliffe spin-coherent state approach which incorporates squeezing effects to analyse semi-classical equations of motion for spin chains with different symmetries. A kink-soliton instability against fluctuations in the complex squeeze parameter is found and quantum corrections to the kink energy are estimated.

1. Introduction

Quasi-one-dimensional magnets are widely used to gain an understanding of non-linear excitations in solids (Lovesey *et al* 1984, Bishop *et al* 1987). In the classical continuum limit some special models have been identified with completely integrable soliton-bearing classical field theories (Tjon and Wright 1977, Fogedby 1980). For other systems where a complete set of solutions has not been found at least special solitary solutions exist (Mikeska 1978, 1980).

To include quantum effects in the description of the latter class of systems semi-classical methods have to be applied. WKB quantisation of the classical solitary excitations can be used to calculate their energy in perturbation theory (in a system of spins of length S the small parameter is $\propto 1/S$) (Mikeska 1982). A different approach neglects the evolution of off-diagonal elements in a basis of states with a well defined classical limit ($S \to \infty$) to general semi-classical equations of motion for the spin chain. Balakrishnan and Bishop (1985) used the spin-coherent states basis introduced by Radcliffe (1971) to obtain soliton solutions for the isotropic Heisenberg spin chain and to calculate the soliton energy–momentum relation within this approach. The extension of their results to a true quantum regime [$S \propto o(1)$] leads to some criticism (Haldane 1986) but in the semi-classical regime the approximations made can be justified.

In this paper we present an extension of Radcliffe's concept of the spin-coherent states as well as of the work of Balakrishnan and Bishop: Introducing an additional parameter which allows for a dynamical squeezing of the spin states we obtain $1/S$ corrections to the classical equations of motion and one further equation for the complex squeeze parameter. Similar approaches have proved to be successful in the field of quantum optics for the description of non-classical properties of light (Pike and Sarkar

† Present address: Department of Physics, University of Virginia, Charlottesville, VA 22901, USA.
‡ Permanent address: Institute of Physics, Warsaw Technical University, Koszykowa 75, 00-662 Warsaw, Poland.

1986) and in the investigation of semi-classical properties of quantum systems that show chaotic behaviour in their classical limit (Frahm and Mikeska 1985).

Our paper is organised as follows: in § 2 we give the definition of the spin-squeezed states and describe their main properties. In §§ 3 and 4 we derive the semi-classical equations of motion for different spin chain models and analyse the stability of the classical soliton solutions against quantum effects. In § 5 we calculate the energy of these excitations to order $1/S$ within our approach to compare our results with others obtained previously.

2. Formulation of the approach

For the description of a quantum spin system of N spins the dynamical equations have to be solved in a Hilbert space of dimension $(2S+1)^N$. However, in the classical limit (namely $S \rightarrow \infty$) each spin can be parametrised by two angles φ, ϑ only:

$$S = \hat{S}\{\cos\varphi \cos\vartheta, \sin\varphi \cos\vartheta, \sin\vartheta\} \tag{2.1}$$

where $\hat{S} = [S(S+1)]^{1/2}$ is the modulus of the spin vector. Thus in the semi-classical regime a reduction of the number of variables should be possible by considering certain quasi-classical states in the Hilbert space only.

A prototype of such a class of quantum states are the well known spin-coherent states (Radcliffe 1971) for a single spin. The classical variables φ and ϑ are used to parametrise an overcomplete set of states

$$|\vartheta, \varphi\rangle = \exp(-\mathrm{i}\varphi S^z) \exp[\mathrm{i}(\vartheta - \pi/2)S^y]\,|S\rangle \tag{2.2}$$

with $S^z|S\rangle = S|S\rangle$ and $0 \leqslant \varphi \leqslant 2\pi$, $-\pi/2 \leqslant \vartheta \leqslant \pi/2$ (we use $\hbar = 1$ throughout this paper). The classical limit within this formulation is achieved by taking diagonal matrix elements only where linear functions of the spin operators give classical expectation values:

$$\langle\vartheta, \varphi|S|\vartheta, \varphi\rangle = S\{\cos\varphi \cos\vartheta, \sin\varphi \cos\vartheta, \sin\vartheta\} \tag{2.3}$$

(the difference in the factors $S/\hat{S}$ in (2.1) and (2.3) disappears in the classical limit $S \rightarrow \infty$).

Diagonal matrix elements taken with non-linear expressions in spin operators give corrections of order $1/S$, e.g.,

$$\langle\vartheta, \varphi|\,(S^z)^2\,|\vartheta, \varphi\rangle = S^2[\sin^2\vartheta + (1/2S)\cos^2\vartheta]. \tag{2.4}$$

These $1/S$ corrections as compared with the equivalent classical expressions give rise to a renormalisation of the constants that appear in the dynamical equations. However, this ansatz gives only a small part of the information on the evolution of the quantum system since the classical angles φ and ϑ are the only degrees of freedom.

For a more comprehensive inclusion of these effects we introduce spin-squeezed states (SSS) in the following where an additional variable is used to take into account the dynamical distortion of the Radcliffe coherent states.

Following Radcliffe's construction of the spin-coherent states we first consider the ground-state squeezing of an harmonic oscillator. In coordinate representation the squeezed state reads $[\mathrm{Re}(\Gamma) > 0]$:

$$\langle x|\Gamma\rangle = [N(\Gamma)]^{-(1/2)} \exp[-(\Gamma/2)x^2] \tag{2.5}$$

where $|\Gamma = 1\rangle \equiv |0\rangle$ is the ground state of the Hamiltonian $H = p^2/2 + x^2/2$. To construct the operator U_Γ that generates $|\Gamma\rangle$ from $|0\rangle$

$$U_\Gamma\,|0\rangle = |\Gamma\rangle \tag{2.6}$$

equation (2.5) has to be expanded into the harmonic oscillator eigenstates:

$$|\Gamma\rangle = \sum_n c_n\,|n\rangle \tag{2.7}$$

where (H_n are the Hermite polynomials):

$$c_n \propto [N(\Gamma)]^{-(1/2)} \int \mathrm{d}x\, H_n(x) \exp\left(-\frac{1+\Gamma}{2}x^2\right). \tag{2.8}$$

From symmetry we have $c_{2n+1} = 0$. The remaining integrals are easily evaluated giving

$$c_{2n} = \frac{1}{n!}\left(N(\Gamma)^{-1}\frac{2\sqrt{\pi}}{1+\Gamma}(2n)!\right)^{1/2}\left(\tfrac{1}{2}\frac{1-\Gamma}{1+\Gamma}\right)^n. \tag{2.9}$$

With the harmonic oscillator raising operator a^+ a closed expression for U_Γ can be found from (2.7):

$$U_\Gamma = [N(\Gamma)]^{-(1/2)} \exp\{\tfrac{1}{2}[(1-\Gamma)/(1+\Gamma)](a^+)^2\}. \tag{2.10}$$

This operator is used in quantum optics for the description of squeezed photon states (see Pike and Sarkar 1986).

We now introduce an analogous ansatz for the squeezing operator for the spin-coherent 'ground state' $|S\rangle$ in the form:

$$U(a) = [N(a)]^{-(1/2)} \exp[(a/4S)(S^-)^2] \tag{2.11}$$

($a \in \mathbb{C}$ is our squeezing parameter) which gives the following for the SSS

$$|a\rangle = U(a)\,|S\rangle = [N(a)]^{-1/2}\sum_0^S \frac{1}{n!}\left(\frac{a}{4S}\right)^n\left(\frac{(2n)!\,(2S)!}{(2S-2n)!}\right)^{1/2}|S-2n\rangle. \tag{2.12}$$

Obviously $a = 0$ corresponds to the normal coherent spin state. The normalisation of $|a\rangle$ gives:

$$N(a) = \sum_0^S \frac{(2n)!}{(n!)^2}\frac{(2S)!}{(2S-2n)!}\left(\frac{|a|}{4S}\right)^{2n}. \tag{2.13}$$

For every finite value of S this is a polynomial in $|a|$. However, since we are especially interested in the semi-classical limit $S \to \infty$ we can give an approximate expression for $N(a)$ which is valid in the case of moderate squeezing. Assuming that the terms in (2.13) vanish sufficiently fast, e.g., $|a| < 1$ we can use $n \ll S$ in each term which yields

$$N_\infty(a) \simeq \sum_0^S \frac{(2n)!}{(n!)^2}\left(\frac{|a|}{2}\right)^{2n} = (1-|a|^2)^{-1/2}. \tag{2.14}$$

Next one has to calculate expectation values of spin operators within the SSS approach. They can be written in terms of derivatives of the generating function (2.13) (or (2.14) in the semi-classical regime) giving e.g.,

$$\langle a|S^z|a\rangle = S - E_1 \qquad \langle a|S^x|a\rangle = \langle a|S^y|a\rangle = 0$$

$$\langle a|(S^z)^2|a\rangle = S^2 - (2S-1)E_1 + E_2$$

$$\langle a|(S^x)^2|a\rangle = [S + (2S-1)E_1 - E_2]/2 + S\,\mathrm{Re}(a)E_1/|a|^2 \tag{2.15}$$

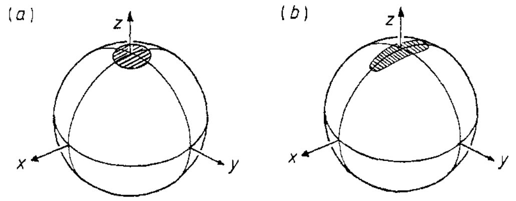

Figure 1. Probability distribution $P(\vartheta, \varphi) = |\langle\vartheta, \varphi|a\rangle|^2$ of the spin-squeezed state (equation (2.12)) for (*a*) $a = 0$ (spin-coherent state) and (*b*) $a > 0$.

$$\langle a|(S^y)^2|a\rangle = [S + (2S - 1)E_1 - E_2]/2 - S\,\mathrm{Re}(a)E_1/|a|^2$$
$$\langle a|S^xS^y + S^yS^x|a\rangle = 2S\,\mathrm{Im}(a)E_1/|a|^2$$

where

$$E_k = (1/N)\,|a|^k(\partial^k/\partial|a|^k)N. \tag{2.16}$$

From equations (2.15) the action of the squeeze parameter a is easily seen: The modulus of a determines the amount of squeezing while different choices of the phase α of a lead to different orientations of the squeezed 'wave-packet' (a change in α by π is equivalent to a rotation of the state about the z-axis by $\pi/2$). In figure 1 the effect of the squeezing operator $U(a)$ is shown by visualisation of the probability distribution of the state $|a\rangle$ on the sphere of accessible spin directions. Note that $\langle a|(S^x)^2|a\rangle\,\langle a|(S^y)^2|a\rangle = S^2/4 + o(|a|^2)$. An equivalent equation holds for the harmonic oscillator squeezed states for arbitrary values of the squeeze parameter (see Pike and Sarkar 1986). The $o(|a|^2)$ corrections as well as the decrease of the expectation value $\langle S^z\rangle$ in the SSS is a consequence of the spherical symmetry of the spin phase space and may be used as a measure for the validity of the description of the quantum system in terms of these states in the semi-classical limit: As soon as E_1 becomes of the order of S (namely the corrections to $\langle S^z\rangle$ become large) more details of the quantum mechanical nature of the system have to be taken into account for a complete description.

All the results given above have been calculated for the SSS (2.12) pointed in z-direction, e.g., $\langle a|\boldsymbol{S}|a\rangle \| \boldsymbol{e}_z$. However, any other state can be obtained by rotation of this one. This gives a three parameter family of quantum states:

$$|\vartheta, \varphi, a\rangle = \exp(-\mathrm{i}\varphi S^z)\exp[\mathrm{i}(\vartheta - \pi/2)S^y]U(a)\,|S\rangle \tag{2.17}$$

which we shall use for the description of semi-classical spin dynamics in the following. Expectation values of spin operators within these states are obtained from (2.15) by rotation, e.g.,

$$\begin{aligned}
&\langle\vartheta, \varphi, a|S^x|\vartheta, \varphi, a\rangle = \langle S^z\rangle_0 \cos\vartheta\cos\varphi \\
&\langle\vartheta, \varphi, a|(S^z)^2|\vartheta, \varphi, a\rangle = \langle(S^z)^2\rangle_0 \sin^2\vartheta + \langle(S^x)^2\rangle_0 \cos^2\vartheta \\
&(\langle\,.\,\rangle_0 = \langle a|\,.\,|a\rangle).
\end{aligned} \tag{2.18}$$

3. Equations of motion for spin chains in the squeezed spin basis

3.1. *Isotropic Heisenberg chain*

We consider now a one-dimensional isotropic quantum spin model described by the following Hamiltonian ($J > 0$)

$$H = -J\sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}. \tag{3.1}$$

The equations of motion for the spin operators read:

$$\mathrm{i}\partial_t S_n^+ = J[(S_{n-1}^z + S_{n+1}^z)S_n^+ - S_n^z(S_{n-1}^+ + S_{n+1}^+)] \tag{3.2a}$$

$$\mathrm{i}\partial_t S_n^z = -(J/2)[(S_{n-1}^- + S_{n+1}^-)S_n^+ - S_n^-(S_{n-1}^+ + S_{n+1}^+)] \tag{3.2b}$$

where the Planck constant has been omitted. Similar to how it was done for Radcliffe spin coherent states (Balakrishnan and Bishop 1985), we make the following ansatz for the basis of spin squeeze states of the spin chain (3.1)

$$|\Psi\rangle = \prod_n |\vartheta_n, \varphi_n, a_n\rangle \tag{3.3}$$

where a triplet of parameters ϑ_n, φ_n, a_n defines a SSS for a single spin operator $\boldsymbol{S}_n$ as described in § 2.

Taking diagonal matrix elements of equations (3.2) in the states (3.3) and going to the continuum limit $\langle S_n\rangle \to \langle S(z)\rangle$ (d being the lattice constant) one obtains

$$\mathrm{i}\partial_t\langle S^+\rangle = Jd^2[(\partial_{zz}\langle S^z\rangle)\langle S^+\rangle - \langle S^z\rangle(\partial_{zz}\langle S^+\rangle)] \tag{3.4}$$

where $\langle\vartheta_n, \varphi_n, a_n|S_n^\alpha|\vartheta_n, \varphi_n, a_n\rangle \to \langle\vartheta(z), \varphi(z), a(z)|S^\alpha(z)|\vartheta(z), \varphi(z), a(z)\rangle = \langle S^\alpha\rangle$ and an analogous equation for $\partial_t\langle S^z\rangle$. Of course the evolution of operator diagonal elements does not describe completely the evolution of the system's wavefunction, which for times $t > 0$ is a superposition of different coherent states even if for $t = 0$ it was represented by a single one. However, since the classical limit of the dynamical equations is completely obtained from the diagonal elements a description of the semi-classical ($S \to \infty$) system within these states should be possible as long as the approach is applicable (namely $E_1(Z) \ll S$, compare with the discussion in § 2). From (3.4), (2.18) and (2.15) dynamical equations for $\varphi(z)$ and $\vartheta(z)$ may be obtained:

$$\cos\vartheta\,\partial_t\varphi = JSd^2\{-(1 - E_1S^{-1})[\partial_{zz}\vartheta + (\partial_z\varphi)^2\sin\vartheta\cos\vartheta] + 2S^{-1}(\partial_zE_1)\partial_z\vartheta\} \tag{3.5a}$$

$$\partial_t\vartheta = JSd^2\{(1 - E_1S^{-1})[\cos\vartheta\,\partial_{zz}\varphi - 2\sin\vartheta(\partial_z\varphi)\partial_z\vartheta] - 2S^{-1}\cos\vartheta(\partial_zE_1)\partial_z\varphi\} \tag{3.5b}$$

where E_1 is given by (2.16). These are just the classical equations of motion for the Heisenberg spin chain with additional terms of the order of S^{-1}. To get an equation for E_1 (which is a function of $|a|$ only) one can use the identity

$$\sum_\alpha \langle(S^\alpha)\rangle^2 = (S - E_1)^2 \tag{3.6}$$

which after time differentiation leads to

$$-(S - E_1)\partial_tE_1 = \langle S^z\rangle\partial_t\langle S^z\rangle + \mathrm{Re}(\langle S^-\rangle\partial_t\langle S^+\rangle). \tag{3.7}$$

Taking into account equations (3.5) and (3.2*b*) we get from (3.7) $\partial_tE_1 = 0$ from which it follows that $|a| =$ const. Thus in the case of the isotropic Heisenberg chain the initial deformation of the local spin states (e.g., squeezing) remains preserved for arbitrary times. When $|a(z)| = \text{const} \neq 0$ ($a = 0$ is the non-squeezed coherent spin state) we have just a renormalisation of a timescale in the classical equations (3.5). We shall see, however, that adding any anisotropy to the spin Hamiltonian leads to $\partial_tE_1 \neq 0$ and thus to a dynamical generation of squeezing of the local states.

3.2. *Anisotropic spin chain in the presence of a magnetic field*

Let us consider now an anisotropic quantum spin chain in the presence of the external magnetic field B:

$$H = -J\sum_n \boldsymbol{S}_n\boldsymbol{S}_{n+1} + A\sum_n (S_n^z)^2 - B\sum_n S_n^x. \tag{3.8}$$

Assuming that $A > 0$ we get an easy plane model that is often used to describe the quasi-one-dimensional ferromagnet $CsNiF_3$ (Mikeska 1978), while the case $A < 0$ and $B = 0$ corresponds to the Ising symmetry. Similar to the isotropic Hamiltonian (3.1) one gets from (3.8) in the continuum limit

$$\partial_t\langle S^+\rangle = \mathrm{i}Jd^2(\langle S^z\rangle\partial_{zz}\langle S^+\rangle - \langle S^+\rangle\partial_{zz}\langle S^z\rangle) + \mathrm{i}A\langle S^zS^+ + S^+S^z\rangle + \mathrm{i}B\langle S^z\rangle$$
$$\partial_t\langle S^z\rangle = (\mathrm{i}Jd^2/2)(\langle S^+\rangle\partial_{zz}\langle S^-\rangle - \langle S^-\rangle\partial_{zz}\langle S^+\rangle) - B\langle S^y\rangle. \tag{3.9}$$

From (3.7) and (3.9) we get after some algebra

$$\begin{aligned}-\cos\vartheta(S - E_1)\partial_t\varphi = {} & Jd^2\{(S - E_1)^2[\vartheta'' + (\varphi')^2\sin\vartheta\cos\vartheta] - 2(S - E_1)E_1'\vartheta'\}\\ & - A\sin(2\vartheta)[(S - \tfrac{1}{2})(S - 3E_1) + \tfrac{3}{2}E_2 - SE_1\,\mathrm{Re}(a)/|a|^2]\\ & - B(S - E_1)\sin\vartheta\cos\vartheta\end{aligned} \tag{3.10}$$

$$\begin{aligned}\partial_t\vartheta = {} & d^2JS\{(1 - E_1S^{-1})[\cos\vartheta\varphi'' - 2\varphi'\vartheta'\sin\vartheta] - 2S^{-1}\cos\vartheta E_1'\varphi'\}\\ & - \mathrm{cotan}\vartheta(S - E_1)^{-1}\partial_tE_1 + B\sin\varphi - 2AS\\ & \times\mathrm{cotan}\vartheta\,\mathrm{Im}(a)\,|a|^{-2}E_1/(S - E_1)\end{aligned} \tag{3.11}$$

and

$$\partial_tE_1 = -2SA\cos^2\vartheta\,E_1\,\mathrm{Im}(a)/|a|^2 \tag{3.12}$$

where E_1 and E_2 are defined by (2.16) and a prime means a spatial derivative. Taking into account a semi-classical expression coming from (2.14) $E_{1(\mathrm{sc})} = |a|^2/(1 - |a|^2)$, we get from (3.12)

$$\partial_t|a| = -AS\cos^2\vartheta(1 - |a|^2)\,\mathrm{Im}(a)/|a|. \tag{3.13}$$

Putting into (3.10–3.11) the expression $E_{1(\mathrm{sc})}$ and the semi-classical formula $E_{2(\mathrm{sc})} = E_{1(\mathrm{sc})} + 3E^2_{1(\mathrm{sc})}$ we see that equations (3.10–3.11) and (3.13) do not form together a complete set of equations because of a complex character of the squeeze parameter a. To obtain an evolution of the phase α of this parameter ($\alpha = \tan^{-1}[\mathrm{Im}(a)/\mathrm{Re}(a)]$) we need to consider one more operator equation. Due to the symmetry of the Hamiltonian (3.8) it seems to be natural to choose $(S^z)^2$ for this operator, however, we have not proved that every other choice would lead to the equivalent final results. From (3.8) we get

$$\begin{aligned}\partial_t\langle(S_n^z)^2\rangle = {} & (J\mathrm{i}/2)\langle(S_n^+S_n^z + S_n^zS_n^+)(S_{n+1}^- + S_{n-1}^-) - (S_n^-S_n^z + S_n^zS_n^-)(S_{n+1}^+ + S_{n-1}^+)\rangle\\ & - B\langle S_n^zS_n^y + S_n^yS_n^z\rangle.\end{aligned} \tag{3.14a}$$

Using (2.15) and (2.18) we obtain in the continuum limit of (3.14a)

$$\begin{aligned}\partial_t\langle(S^z)^2\rangle = {} & -Jd^2[\![\sin\vartheta\cos\vartheta[(2S - 1)(S - 3E_1) + 3E_2 - 2S|a|^{-2}E_1\,\mathrm{Re}(a)]\\ & \times[2E_1'\varphi'\cos\vartheta + 2\vartheta'\varphi'\sin\vartheta(S - E_1) - \varphi''\cos\vartheta(S - E_1)]\\ & - 2S\cos\vartheta\,|a|^{-2}E_1\,\mathrm{Im}(a)\{-\cos\vartheta E_1'' + 2E_1'\vartheta'\sin\vartheta - (S - E_1)[\vartheta''\sin\vartheta\\ & + (\vartheta')^2\cos\vartheta] - (\varphi')^2(S - E_1)\cos\vartheta + 2d^{-2}(S - E_1)\cos\vartheta\}]\!]\\ & - B\{\sin\varphi\sin\vartheta\cos\vartheta[(S - 3E_1)(2S - 1) + 3E_2 - 2SE_1\cos\alpha/|a|]\\ & - \cos\varphi\cos\vartheta\,2SE_1\sin\alpha/|a|\}.\end{aligned} \tag{3.14b}$$

However from (2.15) and (2.18) one also gets

$$\langle(S^z)^2\rangle = [S^2 - (2S - 1)E_1 + E_2]\sin^2\vartheta + \cos^2\vartheta[S - E_2 + (2S - 1)E_1$$

$$+ 2S|a|^{-2}\,\mathrm{Re}(a)E_1]/2. \tag{3.15}$$

After longer algebra, combining (3.14*b*) and (3.15) and taking into account (3.10–3.13) we obtain in the semi-classical limit (2.14) up to the leading terms in S

$$\cos\vartheta\,\partial_t\alpha = 2JSd^2[(\varphi')^2\cos\vartheta + \vartheta''\sin\vartheta + (\vartheta')^2\cos\vartheta - 2d^{-2}\cos\vartheta] - 2AS\cos^3\vartheta[1 + \cos\alpha(1 + |a|^2)/(2|a|)] - 2B\cos\varphi. \tag{3.16}$$

Combining equations (3.13) and (3.16) leads finally to

$$\partial_t a = 2\mathrm{i}\{[(\varphi')^2 + \vartheta''\tan\vartheta + (\vartheta')^2]d^2JS - 2JS - B\cos\varphi/\cos\vartheta\}a - \mathrm{i}AS\cos^2\vartheta(1 + a)^2. \tag{3.17}$$

Now we need to solve the system of equations (3.10–3.11) and (3.17). It is essential to see that neglecting in equations (3.10–3.11) all terms proportional to E_1/S, E_2/S and their derivatives (which is allowed in the semi-classical approximation) we get classical equations of motion for the angles ϑ and φ (compare, for example, with Etrich and Mikeska 1988) with no coupling to the squeeze parameter a. Moreover there are no spatial derivatives of this variable in equation (3.17). In this sense to solve approximately the system of equations (3.10–3.11, 3.17) in the limit of $S \gg 1$ we can use classical solutions of (3.10–3.11) and simply put them into (3.17). For a fixed value of the z-coordinate the last equation becomes an ordinary differential equation for the variable a

$$\partial_t a(z,t) = 2\mathrm{i}\zeta(z,t)a(z,t) - \mathrm{i}\eta(z,t)[1 + a(z,t)]^2 \tag{3.18}$$

where functions $\zeta(z,t) = [(\varphi')^2 + \vartheta''\tan\varphi + (\vartheta')^2]d^2JS - 2JS - B\cos\varphi/\cos\vartheta$ and $\eta(z,t) = AS\cos^2\vartheta$ can be treated as parameters depending only on the particular form of the solutions $\vartheta(z,t)$ and $\varphi(z,t)$ but not on the squeeze variable a. When $\eta(z,t)$ is not identically equal to zero then using the transform $(\partial_t b)/b = (a + 1 - \zeta/\eta)\mathrm{i}\eta$ we get from (3.18)

$$\partial_{tt} b = (\partial_t b)(\partial_t\eta)/\eta + b[\zeta(2\eta - \zeta) - \mathrm{i}(\partial_t\zeta) + \mathrm{i}\zeta(\partial_t\eta)/\eta]. \tag{3.19}$$

Depending on the behaviour of the functions $\zeta(z,t)$ and $\eta(z,t)$ the last equation possesses different kinds of solutions that will be discussed for special cases in the next section.

4. Stability analysis of the squeeze parameter equation of motion

Equation (3.19) becomes much simpler

$$\partial_{tt} b = -\omega^2 b \qquad \omega^2 = \zeta(\zeta - 2\eta) \tag{4.1}$$

if one considers the case of time-independent functions $\zeta(z)$ and $\eta(z)$. This is obviously fulfilled if the solutions for the angles ϑ and φ are static.

(i) Let us assume that $A > 0$ and $B > 0$. The well known exact solution of the classical limit of equations (3.10) and (3.11) has a form of a static sine–Gordon (sG) kink-soliton (Mikeska 1978)

$$\varphi(z,t) = 4\tan^{-1}[\exp(mz)] \qquad \vartheta(z,t) = 0 \tag{4.2a}$$

with $m^2 = B/(JSd^2)$. The corresponding stationary solution of equation (3.18) reads

$$a^{\mathrm{sol}} = -1 - 2\beta\lambda^{-1} + 12\lambda^{-1}/\cosh^2(mz) + \{[-1 - 2\beta\lambda^{-1}$$

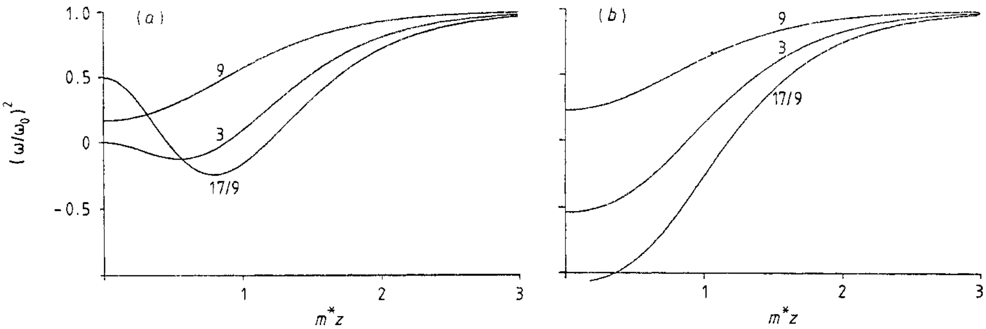

Figure 2. Dependence of the squeeze rotation frequency in presence of a static kink (4.3) on the position along the chain for $\lambda = 3$ (a) and $\lambda = 10$ (b) and different values of β, given on the curves (ω^2 is normalised to its vacuum value, equation (4.4)).

$$+ 12\lambda^{-1}/\cosh^2(mz)]^2 - 1\}^{1/2} \tag{4.2b}$$

where $\beta = 1 + 2JS/B = 1 + 2/(md)^2$ and $\lambda = 2AS/B$ are the parameters that control discreteness- and out-of-plane-effects, respectively. Due to the finite value of the easy-plane anisotropy the solution (4.2a) is stable against meridional (out-of-plane) distortions of the spin components only when $B < B_c = 2AS/3$ (Kumar 1982, Magyari and Thomas 1982, Osano 1984). Using (4.2a) we get from (4.1):

$$\omega^2(z) = B^2[6u(z) - \beta][6u(z) - \beta - \lambda] \tag{4.3}$$

where $u(z) = 1/\cosh^2(mz)$. From equation (4.3) we see that the stability condition $\omega^2(z) > 0$ is automatically fulfilled for $|z| \to \infty$ because

$$\omega_0^2 = \omega^2(\pm\infty) = B^2\beta(\beta + \lambda) \tag{4.4}$$

which corresponds to the stability of the ground state $\boldsymbol{S} = [S, 0, 0]$ of the system (3.1). However as $u \in (0, 1]$ so for $\beta < 6$ in the presence of the kink (4.2a) there is a part of the chain where the spin motion is unstable to fluctuations of the variable b (figure 2). It follows that for the magnetic field higher than $B_c^Q = 2JS/5$ there is an instability in the squeeze parameter, in the sense that $\lim_{t\to\infty}|a| = 1$ for some interval along the z-axis. For $CsNiF_3$, which is often mapped to the Hamiltonian (3.1), there is $A = 5\,k_B$ and $J = 23.6\,k_B$ so for this material the critical field against squeeze parameter fluctuations is higher than the critical field against out-of-plane fluctuations $B_c^Q \simeq 2.8\,B_c$.

For time-dependent functions $\zeta(z, t)$ and $\eta(z, t)$ corresponding to time-dependent solutions $\varphi(z, t)$ and $\vartheta(z, t)$ the analysis of equation (3.19) becomes more complicated. One can however easily see that the quasi-classical solutions of equations (3.10–3.11) in a form of usual low-amplitude spin waves do not lead to the occurrence of any instability in equation (3.19) because in such a case leading terms in this equation come from the stable ground state of the system (3.1).

(ii) We assume now that the Hamiltonian (3.1) possesses an Ising-like symmetry, e.g., we put $A < 0$ and $B = 0$. In such a case the exact static solution of the classical limit of equations (3.10–3.11) has a form of the following kink-soliton:

$$\varphi(z, t) = \varphi_0 = \text{constant} \tag{4.5}$$

$$\vartheta(z, t) = \tan^{-1}\{\sinh[z(2|A|)^{1/2}/(d^2J)^{1/2}]\}. \tag{4.6}$$

This solution is always stable against any small perturbations in ϑ and φ (Tjon and

Wright 1977). Instead of equation (4.3) we now have

$$\omega^2(z) = (2JS)^2[1+\gamma-3\gamma u(z)][1+\gamma-2\gamma u(z)]$$
$$u(z) = 1/\{\cosh^2[(2\gamma)^{1/2}z/d]\} \qquad \gamma = |A|/J. \tag{4.7}$$

Obviously the instability of the b and a parameters occurs for $(1+\gamma)3\gamma < 1$, e.g., for $|A| > |A_c| = J/2$. Results on the absolute stability of the ground state $S = [0, 0, \pm S]$ and on the stability of the small-amplitude spin waves are similar to those for the previously analysed easy-plane chain.

One has to notice that in both considered cases (i) and (ii) the instability of the squeeze parameter appears when the effective kink width is smaller than some critical value. In fact, writing solution (4.2a) as $\varphi/2 = \pi/2 - \tan^{-1}[\sinh(2z/\Delta_1)]$ we get for the critical width $\Delta_1^c = 10^{1/2}d$, while for the solution (4.6) there is a corresponding value $\Delta_2^c = 2d$. Note that the critical values of the kink width are of the order of the lattice constant d and close to the *classical* instability of the static soliton versus discreteness effects (Etrich *et al* 1985). Hence, for a complete understanding of the semi-classical instability in the squeeze parameter and its interplay with the classical one the discrete equations of motion for the system (3.8) should be analysed in the spin-squeezed state approach.

5. Quantum corrections to the soliton energy

For a comparison of the extended spin-coherent approach with different treatments of quantum spin chains in the semi-classical regime we now want to calculate the energy of the semi-classical soliton solution, i.e.,

$$E_S = \langle\psi_{\rm sol}|H|\psi_{\rm sol}\rangle - \langle\psi_{\rm vac}|H|\psi_{\rm vac}\rangle \tag{5.1}$$

with $|\psi_i\rangle = \Pi_n|\varphi_n^i, \vartheta_n^i, a_n^{(i)}\rangle$, $a_n^{\rm sol}$ as given by equation (4.2b) and

$$a_n^{\rm vac} \equiv -(1+2\beta/\lambda) + [(1+2\beta/\lambda)^2 - 1]^{1/2} \tag{5.2}$$

being the corresponding solution for the classical vacuum state (β and λ as given in § 4). Performing the continuum limit and taking corrections of order $1/S$ only we find for the easy-plane anisotropic chain (equation (3.8) with $A > 0$):

$$\langle\psi_i|H|\psi_i\rangle = \frac{BS}{d}\int \mathrm{d}z\,\{(1/m)^2(1-2E_1/S)(\tfrac{1}{2}\varphi_z^2 - 1/d^2)$$
$$+ (\lambda/2S)[\tfrac{1}{2} + E_1(1+\mathrm{Re}(a)/|a|^2)] - (1-E_1/S)\cos\varphi\} \tag{5.3}$$

for solutions with $\vartheta \equiv 0$ which we consider here. The $1/S$ corrections in φ (see equation (3.10)) contribute to second order in $1/S$ only since the classical soliton solution minimises the energy functional. With (5.3) the energy of the static sG soliton is given to o$(1/S)$ by (the classical soliton energy is $E_S^{\rm cl} = 8S(JBS)^{1/2}$):

$$E_S = \langle H_{\rm sol}\rangle - \langle H_{\rm vac}\rangle = E_S^{\rm cl}[1 + (1/8S)(I_s - I_0)] \tag{5.4}$$

where

$$I_s - I_0 = \int \mathrm{d}\xi\,[(\beta+\lambda/2)(E_1^{\rm sol} - E_1^{\rm vac}) + (\lambda/2)(E_1^{\rm sol}/a^{\rm sol} - E_1^{\rm vac}/a^{\rm vac})$$
$$- 6E_1^{\rm sol}\,\mathrm{sech}^2\xi] \tag{5.5}$$

($\xi = mz$). We will now consider this expression in several limiting cases for the parameters λ and β:

(i) $\beta \gg \lambda$, i.e., $a^{(i)} \ll 1$: In this case E_1 may be replaced by a^2 and we have from equations (4.2*b*) and (5.2):

$$a^{\text{vac}} \simeq -\tfrac{1}{2}(1 + 2\beta/\lambda)^{-1}$$
$$a^{\text{sol}} \simeq -\tfrac{1}{2}[1 + 2\beta/\lambda - (12/\lambda)\operatorname{sech}^2\xi]^{-1} \simeq a^{\text{vac}}[1 - (24/\lambda)a^{\text{vac}}\operatorname{sech}^2\xi]. \tag{5.6}$$

Inserting this in the expression for the semi-classical soliton energy (5.4) we find:

$$E_{\text{S}} \simeq E_{\text{S}}^{\text{cl}}[1 - (3/64S)(\lambda/\beta)^2(1 - \lambda/\beta)]. \tag{5.7}$$

(ii) $S^2 \gg \lambda/\beta \gg 1$ (planar limit). (Note that the limit $\lambda/\beta \to \infty$ requires the limit $S \to \infty$ to be taken first because otherwise the classical vacuum solution becomes unstable, i.e., $E_1 \propto o(S)$.) For this range of parameters we have $a^{(i)} = -1 + \varepsilon$, which gives

$$E_1 \simeq 1/2\varepsilon \qquad E_1/a \simeq -1/2\varepsilon = -E_1. \tag{5.8}$$

Using $\lambda/\beta \gg 1$ now we find

$$a^{\text{vac}} = -1 + 2(\beta/\lambda)^{1/2} \tag{5.9a}$$

$$a^{\text{sol}} = -1 + 2[(\beta - 6\operatorname{sech}^2\xi)/\lambda]^{1/2} \tag{5.9b}$$

giving the following result for the renormalised energy of the quantum soliton:

$$E_{\text{S}} = E_{\text{S}}^{\text{cl}}[1 + g^2 f(\beta)] \tag{5.10}$$

where $g^2 = (2A/JS^2)^{1/2} = [2\lambda/(\beta - 1)]^{1/2}/S$ is the relevant parameter for the effects of quantum fluctuations in the planar regime (Mikeska 1982) and

$$f(\beta) = \tfrac{1}{32}[(\beta - 1)/2]^{1/2} \int \mathrm{d}\xi\, [\surd\beta - (\beta - 6\operatorname{sech}^2\xi)^{1/2}]. \tag{5.11}$$

This expression may be evaluated for different values of β (which controls the influence of lattice effects in our approach).

(i) In the continuum limit ($\beta \to \infty$) (5.10) gives:

$$E_{\text{S}} = E_{\text{S}}^{\text{cl}}[1 - (3/16\surd 2)g^2]. \tag{5.12}$$

(ii) For $\beta = 6$, i.e., the value where the semi-classical instability (see § 4) occurs, the renormalised soliton energy is given by

$$E_{\text{S}} = E_{\text{S}}^{\text{cl}}[1 - (\surd 15/16)(\ln 2)\, g^2]. \tag{5.13}$$

(iii) For intermediate values of β equation (5.11) can be integrated numerically, where a monotonous decrease of f with β is found.

Equations (5.12) and (5.13) can now be compared with the results of Mikeska (1982) who has investigated the system (3.10) by means of WKB quantisation of the classical soliton solution in a corresponding field theory. Using an expansion about the planar continuum limit he found for the renormalised soliton energy in the semi-classical limit $E = E_{\text{S}}^{\text{cl}}\{1 - (g^2/8\pi)[1 - o(\lambda)]\}$. Comparing this result with equation (5.12) which should be valid in the same range of parameters we find that our estimate of the quantum corrections is larger by a factor of ~3.3 than that given by Mikeska. This discrepancy may be a consequence of differences in the way the various limiting processes ($S \to \infty$, $\lambda \to \infty$, $\beta \to \infty$) are performed to obtain equation (5.12) and in the field-theoretical treatment by Mikeska. While the dependence on the single dimensionless parameter g^2 in equation (5.12) is the result of a limiting process controlled by $S^2 \gg \lambda/\beta \gg 1$ no such restriction had to be implied explicitly in Mikeska's work. On the other hand, the field-theoretical result might contain contributions by modes neglected in the squeezed-state approach presented in this work.

6. Summary and discussion

The approach developed in the present work forms an extension of the Radcliffe spin-coherent states method by introduction of spin-squeezed states analogous to squeezed states of a quantum harmonic oscillator. As a result in the quasi-classical limit one obtains a system of three coupled equations describing the evolution of a spin vector direction and a value of the complex squeeze parameter a. The equations for the spherical angles possess a classical form with S^{-1} corrections coming from the non-zero values of the parameter a. The squeezing causes the variances $\langle a|(S^x)^2|a\rangle$ and $\langle a|(S^y)^2|a\rangle$ of spin components perpendicular to the quantisation axis to be different from the standard coherent state value $S/2$. For small values of the squeeze parameter the product of these variances is a constant as for harmonic oscillator squeezed states. The spin squeezing also influences (diminishes) the expected value of the S^z component and its quadrature.

The 'ansatz' has been applied to spin chains with different symmetries using as a basis a set of products of single spin-squeezed states. We have shown that an isotropic Heisenberg chain in the presence of a magnetic field is characterised by a rotation of a phase of the squeeze parameter (besides the usual Larmor precession). No changes in the modulus of this parameter are present in such a case. The squeeze phase rotation occurs even for the state of a uniform magnetisation of the isotropic Heisenberg chain.

Very interesting effects occur for an anisotropic Heisenberg chain where the modulus of the squeeze parameter is not a constant of motion. Using the equation for $\partial_t\langle(S^z)^2\rangle$ as an additional evolution equation we obtained a complete set of equations for the spherical angles and the squeeze parameter. We carefully examined the evolution of this parameter when the spherical angles describe the static kink-soliton solution. For the easy-plane ferromagnet with the in-plane magnetic field there is an instability of the squeeze parameter for spins belonging to some part of the chain in the neighbourhood of a kink centre. This instability occurs when the field is too high compared with the value of the exchange constant and it is independent of the classical out-of-plane instability. For the classically stable kinks in an easy-axis chain (with no magnetic field) a similar squeeze instability occurs when the anisotropy parameter is too large compared with the exchange constant. Thus in both of these cases the instability occurs when kinks are too narrow compared with a lattice constant value. No instability occurs for the classical ground states and for the low-amplitude spin waves in these chains.

In addition our approach allows for a calculation of quantum corrections to the soliton energy that were in part studied before for an easy-plane ferromagnet using the WKB method (Mikeska 1982). Depending on the ratios of the anisotropy and exchange parameters to the magnitude of the in-plane magnetic field we got several analytical results for these corrections. A certain difference between some of our results and those obtained by WKB method can be understood as a consequence of the different ways the continuum limits were performed.

We leave open the question on the possibilities of an experimental generation and detection of the spin-squeezed states (and effects connected with them) by just mentioning interesting experiments on squeezed states of light (Pike and Sarkar 1986) performed in recent years.

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